The $\dfrac{LN-M^2}{EG-F^2}$ = Determinant of II Relative to I Formula

Orthonormal basis for tangent space ($=$ span(S_u, S_v))

$\vec{v}_1 = S_u / \| S_u \| = \left(\frac{1}{\sqrt{E}}\right) S_u$

and

$\vec{v}_2 = \left(S_v - \frac{F}{E} S_u\right) / \| S_v - \frac{F}{E} S_u \| = (E S_v - F S_u) / \| E S_v - F S_u \|$

$$= \frac{1}{(E^2 G + F^2 E - 2F^2 E)^{1/2}} (E S_v - F S_u) = \frac{1}{\sqrt{E}} \frac{1}{\sqrt{EG-F^2}} (E S_v - F S_u)$$

$II(\vec{v}_1, \vec{v}_1) = L_{11} / E \qquad II(\vec{v}_2, \vec{v}_2) = \dfrac{1}{E(EG-F^2)} (E^2 L_{22} - 2EF L_{12} + F^2 L_{11})$

$II(\vec{v}_1, \vec{v}_2) = \dfrac{1}{E} \dfrac{1}{\sqrt{EG-F^2}} (E L_{12} - F L_{11})$

$II(\vec{v}_1, \vec{v}_1)\, II(\vec{v}_2, \vec{v}_2) = \dfrac{1}{E^2} \dfrac{1}{EG-F^2} (E^2 L_{11} L_{22} - 2EF L_{11} L_{12} + F^2 L_{11}^2)$

$-II(\vec{v}_1, \vec{v}_2)^2 = \dfrac{1}{E^2} \dfrac{1}{EG-F^2} (-E^2 L_{12}^2 + 2EF L_{11} L_{12} - F^2 L_{11}^2)$

$II(\vec{v}_1, \vec{v}_1)\, II(\vec{v}_2, \vec{v}_2) - II(\vec{v}_1, \vec{v}_2)^2 = \dfrac{1}{E^2} \dfrac{1}{EG-F^2} (E^2 L_{11} L_{22} - E^2 L_{12}^2)$

$$= \frac{L_{11} L_{22} - L_{12}^2}{EG-F^2} = \frac{LN-M^2}{EG-F^2}$$

Fundamental Example for Differential Geometry:

V = tangent space = span(S_u, S_v)
Inner product $\langle\, , \rangle$ on V ← "First fundamental form" I
$\langle aS_u + bS_v, cS_u + dS_v \rangle$
$= acE + (bc+ad)F + bdG$
$Q(aS_u + bS_v, cS_u + dS_v) =$
$= ac\,L_{11} + (bc+ad)L_{12} + bd\,L_{22}.$
"Second fundamental form" $\underline{\mathrm{II}}$

Definition: Trace of Q = <u>mean</u> <u>curvature</u>
Determinant of Q = <u>Gauss</u> <u>curvature</u>

Formulas: Find orthonormal basis by Gram-Schmidt of S_u, S_v:

$$\vec{v}_1 = S_u/\|S_u\| = S_u/\sqrt{E}$$

$$\vec{v}_2 = [S_v - (F/E)S_u]/[S_v - (F/E)S_u] = \sqrt{\frac{E}{EG-F^2}}\,[S_v - (\tfrac{F}{E})S_u]$$

So trace $= Q\left(\frac{S_u}{\sqrt{E}}\right) + Q\left(\sqrt{\frac{E}{EG-F^2}}\,[S_v - \left(\frac{F}{E}\right)S_u]\right)$

$$= \frac{1}{E}L_{11} + \frac{E}{EG-F^2}\left(L_{22} - 2\frac{F}{E}L_{12} + \frac{F^2}{E^2}L_{11}\right)$$

$$= \frac{1}{EG-F^2}\left(\frac{EG-F^2}{E}L_{11} + EL_{22} - 2FL_{12} + \frac{F^2}{E}L_{11}\right)$$

$$= \frac{1}{EG-F^2}\left(GL_{11} + EL_{22} - 2FL_{12}\right)$$

Exercise: determinant $= \dfrac{L_{11}L_{22} - L_{12}^2}{EG-F^2}$